Fermi National Accelerator Laboratory

Cosmic Strings and Superconducting Cosmic Strings [1]

Edmund Copeland

NASA/Fermilab Astrophysics Center
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL. 60510 USA

Abstract

In these lectures I discuss the possible consequences of forming cosmic strings and superconducting cosmic strings in the early universe. Lecture 1 describes the group theoretic reasons for and the field theoretic reasons why cosmic strings can form in spontaneously broken gauge theories. Lecture 2 discusses the accretion of matter onto string loops, emphasising the scenario with a cold dark matter dominated universe. In lecture 3 superconducting cosmic strings are discussed, as is a mechanism which leads to the formation of structure from such strings.

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[1]Lectures presented at the Second Erice Summer School on Dark Matter, May 1988

1 Lecture 1: Cosmic Strings and Phase Transitions in the Early Universe

Recently, astronomers have been systematically probing the very large scale structure of our universe, and have made observations that theorists are now trying to explain. For example, what causal mechanism could produce the primordial energy density perturbations which are thought of as necessary to seed galaxies and clusters of galaxies? How could the inhomogeneity represented by this large scale structure and the galaxies be reconciled with the observed smoothness of the microwave background radiation? The interface of particle physics and cosmology has provided us with one of the most intriguing possible solutions to this problem. The idea that physical processes occurring just 10^{-35} seconds after the initial big bang should directly determine the structures being observed some 15 billion years later is staggering in the extreme. Yet this is just what two recent particle physics ideas, used to determine the initial density fluctuations, suggest. The first is based on quantum fluctuations which arise in Inflationary universe models[1,2]. I will not discuss these models, but strongly advise anyone interested in cosmology to refer to the excellent review articles already written on the subject. Albrechts' article[2] gives a very clear physical picture for the reasons behind inflation. The second is based on cosmic strings [3], and a review of their properties is the goal of these lectures.

In this lecture I will discuss how the strings form and how they evolve in an expanding universe. In lecture 2, I will describe how initial density perturbations grow around long strings and string loops emphasising the Cold Dark Matter (CDM) scenario. In my final lecture I will discuss a very exciting model due to Witten[4], who realised strings may be superconducting in that they can carry persistent currents. This model was later used to account for the formation of the large scale structure in our universe, although rather than acting as seeds around which matter can accrete, these superconducting cosmic strings act as seeds for explosions as they emit electromagnetic radiation, forming bubbles of plasma, on the shells of which the galaxies form and fragment[5]. We will see how the scenario is restricted by the dynamics of such loops, indeed to the extent that it appears the scenario works only for a limited range of coupling constants.

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In motivating the role of particle physics in modern cosmology, I refer the reader to some of the excellent review articles written on the subject[6]. The hot big bang theory of the early universe successfully predicts the Hubble expansion, the microwave background radiation and the light element abundances. It appears to fit in nicely with ideas of particle physics where, as the energy is increased so is the degree of symmetry used to describe the particle interactions. At high enough energies (i.e the very early universe when the temperature was very hot) we find the universe in the state of maximum symmetry. This is spontaneously broken as the universe expands and cools through some critical temperature. However what is required is some source of the perturbations essential to produce the structures we see today. Amongst the more unusual large scale features are giant 'filaments' (linear overdense regions in the galaxy distribution, about $200h_{50}^{-1}$ Megaparsecs long and $10h_{50}^{-1}$Mpc across[7], large 'voids', empty of bright galaxies, $120h_{50}^{-1}$Mpc in diameter, and galaxies lying on the surface of 'bubbles' $40\text{-}60h_{50}^{-1}$Mpc across[8], (h_{50} is Hubbles constant in units of $50kms^{-1}$). An Abell cluster is defined to be a region smaller than $3h_{50}^{-1}$Mpc in radius containing more than 50 bright galaxies. These clusters appear to be clustered on scales of $100h_{50}^{-1}$Mpc with a mean separation of $110h_{50}^{-1}$Mpc.

Now it appears that gravity alone could not have moved galaxies and led to such large scale structure since the big bang. Turok[9] explains this in a succinct argument which I will follow here. Peculiar velocities (velocities relative to the Hubble flow) grow as $t^{\frac{1}{3}}$ in an expanding universe. As we shall see in the next lecture, in the linear regime, $\frac{\delta\rho}{\rho} < 1$, there is a precise relation, $\delta r = H_0^{-1}\delta v$ where δr is the peculiar displacement, δv is the peculiar velocity and the Hubble radius H_0^{-1}, (characterising the expansion rate of the universe)$\equiv 6000h_{50}^{-1}$Mpc. Observational limits placed on the galaxy peculiar velocities are $\delta v_{galaxy} < 600kms^{-1}$, which implies $\delta r \equiv 12h_{50}^{-1}$Mpc, yet structures form on scales some 20 times larger than that. It appears then that by investigating these large scale structures we are looking directly at the primordial density perturbations.

It is generally believed that the early universe was characterised by a series of phase transitions, during which a Higgs field ϕ tended to fall towards the minima of

its potential. As an example, in $\lambda\phi^4$ theory, the lagrangian is given by:

$$\mathcal{L} = \frac{1}{2} \mid \partial_\mu\phi \mid^2 + \frac{1}{2}m_0^2 \mid \phi \mid^2 - \frac{\lambda}{4!} \mid \phi \mid^4 \qquad , m_0^2 > 0 \tag{1}$$

This is the familiar Mexican hat potential. At $T > T_c$, the critical temperature, the fields are in the symmetric 'false vacuum' phase with $< \phi >= 0$. As the universe cools and expands through T_c the ϕ field rolls to the bottom of the potential developing an expectation value $< \phi >^2 = \frac{6m_0^2}{\lambda}$, thereby breaking the symmetry. In fact it is possible to show[19] that the effective mass of the scalar field vanishes at T_c with

$$T_c^2 = 24\frac{m_0^2}{\lambda}$$
$$or$$
$$m^2(T) = m_0^2[1 - \frac{T^2}{T_c^2}] \tag{2}$$

We will now evaluate the spatial correlation of the ϕ field, as it determines the scale of fluctuations in the field[32]. The minimum work required to bring the system out of equilibrium for constant pressure and temperature is the difference in the free energy, between the two vacuum states, ΔF, with a corresponding fluctuation probability $w \propto exp(-\beta\Delta F)$

We concentrate on long wavelength fluctuations in which the ϕ field varies slowly across space. These fluctuations increase anomalously near the transition point. For the case of fluctuations in the symmetric phase, $< \phi >= 0, \Delta\phi = \phi$, then to $O(\phi^2)$ we find that the change in the free energy is (from (1))

$$\Delta F = \int [\frac{m^2(T)}{2} \mid \phi \mid^2 + \frac{1}{2} \mid \partial_\mu\phi \mid^2]d^3x \tag{3}$$

Now expanding $\phi(r)$ as a Fourier series in a volume V
$\phi = \sum_k \phi_k e^{ik.r} \quad , \phi_{-k} = \phi_k^*$
we find

$$\Delta F = \frac{V}{2}\sum_k[k^2 + m^2(T)] \mid \phi_k \mid^2 \ hence \qquad <\mid \phi_k \mid^2>= \frac{T}{V(k^2 + m^2(T))} \tag{4}$$

Note as $T \to T_c$ the long wavelength fluctuations increase. Writing the correlation function as

$$G(r) =< \phi(r_1)\phi(r_2) >, \quad r = r_1 - r_2 \tag{5}$$

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we use

$$G(r) = \sum_k <| \phi_k |^2> e^{ik.r}$$

$$= \int <| \phi_k |^2> e^{ik.r} V \frac{d^3k}{(2\pi)^3} \tag{6}$$

Substituting (4) into (6) we obtain:

$$G(r) \simeq \frac{T_c}{4\pi r} exp(-\frac{r}{\xi}), \quad r \gg \xi \tag{7}$$

where

$$\xi^{-1} = m(T) \simeq \sqrt{\lambda} |< \phi >| \tag{8}$$

is the correlation length.[2] This implies that domains form of size $\xi \sim m^{-1}$ inside which ϕ is correlated, but outside of which there are no correlations. However as $T \to T_c$, $m \to 0$, $\xi \to \infty$, it appears the domains vanish as all the fields become correlated. There is however an upper bound to the correlation length; from cosmology $\xi < H_0^{-1}(t) \sim t$, the distance over which microscopical forces can establish correlations in one Hubble expansion time. In fact there is a tighter constraint which sets the scale over which the domains form. As the domains form and $< \phi > \sim 0$, there is a possibility that thermal fluctuations in the ϕ field could cause ϕ to return to its false vacuum value, hence wipe out the domains. The free energy associated with such a fluctuation with scale ξ is, using the free energy per unit volume f,

$$(2\xi)^3 \Delta f \simeq m^{-3}(T) \frac{m^4(T)}{\lambda}$$

$$\sim \frac{m(T)}{\lambda} \tag{9}$$

The fluctuation has a high probability so long as the free energy required is $\ll$ thermal energy available (T). The two are equal when $\frac{m(T)}{\lambda} \sim T$

$$\frac{1}{T^2} - \frac{1}{T_c^2} \sim \frac{\lambda^2}{m_0^2},$$

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[2]If gauge fields are present, as in superconductors, there is another relevant length which determines the spatial correlations between the fields. It is the London penetration depth and defines the distance the $\underline{B}$ field penetrates the surface of the superconductor. $\delta^{-1} = e |< \phi >| \simeq m_v(T)$. This scale will prove important later on.

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(i.e. for small λ, domains set in when $T \sim T_c$). In fact we can then see that at thermal equilibrium: $[\xi \sim [\sqrt{\lambda} < \phi >]^{-1} \sim [\lambda T_c]^{-1}$ which is the Ginzburg length found in superconductivity.

Hopefully the above arguments are convincing enough to suggest to you that it is worth investigating models of the early universe. One such model was proposed by Kibble[3] in a paper that still remains one of the clearest in the field. At very early times the universe was very hot and the fields describing interactions were in a highly symmetric phase. However as the universe expanded and cooled, symmetry breaking processes would spontaneously occur, occasionally leaving behind remnants of the old symmetric phase, (topological defects), possibly in the form of one dimensional strings or vortex lines[10], two dimensional domain walls or more likely three dimensional monopole configurations[11]. In fact monopoles will always be produced in a GUT symmetry breaking scheme and this is what is commonly referred to as the 'monopole problem'. We don't see any monopoles today, however they would have been produced in a great number density in the early universe. As they are topologically stable, their only means of decay is through annihilation or possibly gravitational radiation[12]. These processes are too slow to rid us of all the monopoles today, so how can we reconcile these apparent discrepancies in the theories? Actually, although I won't go into further details, a possible method of eliminating the monopoles without resorting to inflating them away does exist; inflation would also wipe out any cosmic strings that had been produced prior to the inflationary period, which is bad news for the cosmic strings. It is possible to have a sequence of symmetry breakings which first produce monopoles and then attaches monopole/antimonopole pairs via strings[13]. The monopole flux is confined to exist only on the string, and they rapidly come to annihilate.

Returning to the condition for the existence of these topological defects[3], consider a gauge theory with a symmetry group G, this is the group whose elements leave the full potential V, invariant when acting on it. In the phase transition this group is broken to a sub group H as the fields pick up expectation values. H contains the elements of the original group G which when acting on the fields ϕ, leave them with their expectation value. In fact the manifold of degenerate vacuum states (the manifold corresponding to the state of least energy in the theory), is identified with

the coset space: $M = \frac{G}{H}$.[3] What then does the topology of these various coset spaces look like? For the answer we look to group theory. The condition for the existence of strings is that the first homotopy group $\Pi_1(\frac{G}{H}) \equiv \Pi_1(M)$ be non trivial. The vacuum manifold M must contain non contractable loops. We will look at a specific example later, a more general class of models which contain non contractable loops has been established including some based on superstring theories[14]. Strings are not the only defects that form of course. In fact if $\frac{G}{H}$ is disconnected, then $\Pi_0(M)$ is non trivial and wall like defects form where $V \neq V_{min}$ inside the wall. If $\frac{G}{H}$ contains non contractable 2- spheres then $\Pi_2(M) \neq 1$ and the resulting defect is a monopole.

For the rest of this lecture I will concentrate on string like discontinuities. The most familiar strings, i.e flux tubes in superconductors correspond to the complete breaking of an Abelian group G=U(1)[10]. The lagrangian for the theory is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} \mid D_\mu\phi \mid^2 - \frac{\lambda}{4!}(\mid \phi \mid^2 - \eta^2)^2 \tag{10}$$

where $F_{\mu\nu} = d_{[\mu}A_{\nu]}, D_\mu = \partial_\mu + ieA_\mu, A_\mu$ is the gauge field, e the gauge coupling constant, λ is the self coupling of the Higgs field and η is the value of the symmetry breaking Higgs field. The manifold M of ground states is a circle
$M = [\phi \mid \phi = \eta e^{i\theta}, 0 \leq \theta \leq 2\pi]$
i.e V is minimised by $\phi = \eta e^{i\theta}$ with θ arbitrary, corresponding to the winding number which is an integer. We have already seen how domains could be formed during a phase transition. Now as the system cools below the critical temperature, $(T_c \sim \eta)$, the ϕ field begins to fall to the minima of it's potential. Domains form of size $\xi \sim \eta^{-1}$ due to the thermal fluctuations of the ϕ field. In these domains ϕ points in arbitrary directions in M, but match smoothly at the boundary, with θ varying so as to cause defects to form on the edges common to certain domains. This is easily seen. Consider one such edge where θ varies by 2π in encircling the edge, i.e all around the edge we continuously encircle V_{min}. This implies that ϕ must vanish on that edge for it corresponds to a region where it is not in V_{min}. Such regions line up to minimise the spatial gradient energy, forming a defect line or cosmic string. It corresponds to a thin tube of vacuum energy, V(0), being stored in there. These lines where $\phi = 0$ are

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[3]For example, if (1) $G \equiv Z_2$, (i.e $V(\phi)$ invariant under $\phi \longrightarrow -\phi$), $H = 1, \Longrightarrow V_{min} = \frac{G}{H} = Z_2$
(2) $G = U(1)$, (i.e $\phi \longrightarrow e^{i\theta}\phi$), $H = 1, \Longrightarrow V_{min} = \frac{G}{H} = U(1)$
(3) $G = SO(3)$, (i.e $\phi \longrightarrow O_{ab}\phi_b$), $H = SO(2), \Longrightarrow V_{min} = \frac{G}{H} = S_2$

either in the form of closed loops or infinitely long, for if they had ends, then it would be possible to move the circle (corresponding to V_{min}) beyond the end of the string and then shrink it continuously to a point without having to encounter the $\phi = 0$ region. Hence $\Pi_1(M) = 1$ in that region, there would then be contractable loops and the strings wouldn't exist [4].

Returning to the string solutions of (10), we look for z- independent static solutions to the field equations[10]. Figure (1) shows the resulting solutions for $\mid \phi \mid$ and $\underline{B} = \underline{\nabla} \wedge \underline{A}$ as a function of the radial distance from the string. The width of the string is roughly $m_\phi^{-1} \sim (\sqrt{\lambda}\eta)^{-1}$ where m_ϕ is the Higgs mass. The string tension, or mass per unit length:

$$\mu = \int d^2 r [\frac{1}{2} \mid (\underline{\nabla} + ie\underline{A})\phi \mid^2 + \frac{\lambda}{4!}(\mid \phi \mid^2 - \eta^2)^2 + \frac{1}{2}\underline{B}^2] \tag{11}$$

hence from $V(\phi)$:

$$\mu \sim (\sqrt{\lambda}\eta)^{-2}\lambda\eta^4$$
$$\sim \eta^2 \tag{12}$$

For example if the symmetry breaking scale is during the GUT era: $[\eta \sim 10^{15}$ or $10^{16} Gev]$ then the dimensionless parameter $G\mu$, (G is Newtons constant), lies between 10^{-7} and 10^{-5}. As we shall see observational constraints[15] place a tight upper bound, (not lower) on $G\mu$ of: $[G\mu \leq 10^{-5}]$. There is a big difference between global and local strings. The latter, as the name implies possess a local gauge field whose presence results in no long range interactions between the Higgs fields: $[\lim_{r \to \infty}(\underline{\nabla} + ie\underline{A})\phi = 0]$. Also the magnetic flux in such strings is quantised in units of $\frac{2\pi}{e}$: $[\int \underline{B}.d\underline{S} = \oint \underline{A}.d\underline{l} = \frac{2\pi}{e}]$ where $d\underline{S}$ and $d\underline{l}$ are the area and line elements surrounding a portion of string. If a global symmetry is broken, there are no local gauge fields present, resulting in Goldstone bosons, long range forces and an infinite string mass per unit length. Most of the work on strings has involved local strings, apart from one important result we will come to soon.

The usual method for numerically forming cosmic strings is referred to as the Kibble mechanism. A lattice of domains is constructed typically of size $\sim \xi$, (the

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[4]Strings could be finite in length, by connecting them to monopoles or domain walls[13]

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length scale above which the orientation of the Higgs fields are uncorrelated). In each domain a value of $\phi \in V_{min}$ is randomly chosen , as this reflects ϕ choosing a minimum energy configuration as the phase transition is passed through. After each domain has a value in it, look at each link on the lattice and using a prescription to smoothly vary the phases from one domain to the next, decide if there is a net winding number, anti winding number or no net winding number, hence is there a string, anti string or no string passing along that link. Numerical tests of this mechanism[16] indicate that after the phase transition, about 80% of the string is in long 'infinite' string as long as the box in which the simulation is performed. The rest is in a scale invariant distribution of closed loops where the number of loops between radius r and r + dr,

$$n(r) \propto \frac{dr}{r^4}$$

(i.e independent of ξ). At high densities both the infinite and closed loops of string are in the form of Brownian walks of length $L \sim \frac{r^2}{\xi}$. More recently analytical approaches have placed these predictions on firmer ground. First of all Mitchell and Turok[17], by counting the density of states in the quantised closed bosonic string, demonstrated all the above results. Recently David Haws, Ray Rivers and myself[18] using finite temperature field theory, have investigated the distribution of the Higgs field around the phase transition, also deriving the same behaviour. We are unable to derive a precise number for the amount of infinite string length (i.e 80%), because that is a phenomenon out of equilibrium where the canonical (and microcanonical) ensembles break down. I will spend a little while explaining our technique, although I will also comment on the regimes where the two results can and can not be compared.

The simplest theory to possess vortex solutions is scalar QED

$$\mathcal{L}[\phi, A] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} \mid (\partial_\mu + ieA_\mu)\phi \mid^2 + \frac{1}{2}m_0^2 \mid \phi \mid^2 - \frac{\lambda}{4!} \mid \phi \mid^4, \quad m_0^2 > 0 (13)$$

The partition function is:

$Z \propto \int D\phi D\phi^* DA (det M) exp(-I_\beta[\phi, A])$

where det M describes the gauge fixing, and

$I_\beta[\phi, A] = -\int_0^\beta d\tau \int d^3x \mathcal{L}_E[\phi, A]$

with $\mathcal{L}_E$ the Euclidean form of the lagrangian, (i.e write $\phi[x, \tau] = \sum_n \phi_n(x) exp(\frac{2\pi i n\tau}{\beta})$)
[5]. Now to $O(\lambda, e^2)$, for $\frac{m}{T} \ll 1$ we find the one loop correction to the free energy,

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[5]It is important to note that we are not using the mean field approach here, in that case we would expand about constant solutions for ϕ and A

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having integrated out the massive modes:

$V(\phi) = -\frac{1}{2}m_0^2 \mid \phi_0 \mid^2 [1 - \frac{T^2}{T_c^2}] + \frac{\lambda}{4!} \mid \phi_0 \mid^4$

where $T_c^2 = \frac{m_0^2}{\frac{\lambda}{18} + \frac{e^2}{4}}$. This is then substituted back into the partition function leaving only the functional integration over the zero modes remaining.

$Z = \int D\phi_0 D A_{0\mu} exp(-\beta \bar{I}[\phi_0, A_{0\mu}])$

The statistical properties of strings around the phase transition are found by evaluating Z with string like configurations. The dominant contributions to the integral come from field configurations satisfying the stationary equations $\frac{\partial I}{\partial \phi} \mid_{\phi=\phi_{saddle}, A^\mu = A^\mu_{saddle}} = 0$, $\frac{\partial I}{\partial A_\mu} \mid_{\phi=\phi_{saddle}, A^\mu = A^\mu_{saddle}} = 0$.

$$\partial^i F_{ji} = \frac{1}{2} ie(\phi^* \partial_j \phi - \phi \partial_j \phi^*) - e^2 A_j \mid \phi \mid^2$$

$$(\partial_i + ie A_i)^2 \phi = -m_0^2(1 - \frac{T^2}{T_c^2})\phi + \frac{\lambda}{3!} \mid \phi \mid^2 \phi$$

The contribution of any solution to Z is found by substitution. The solution $\phi = const, A = 0$ is a solution of minimum energy, so gives the maximum contribution. Well away from T_c, Z is well represented by this term. However as $T \to T_c$ we must sum all maxima to the functional because the total contribution due to the large number of non constant field configurations becomes larger. This is easy to see, as $T \to T_c$, the mass, $m(T) \to 0$, so it becomes possible to form strings at no energetic cost, and there is a second order phase transition. There are many string configurations and we must sum over them all[10],[6] here though the solutions vary with temperature.[7] In particular

$\lim_{r \to \infty} \mid \phi(r) \mid \longrightarrow \eta, \qquad \eta^2 = \frac{m_s^2}{\lambda}, \qquad m_s^2 = m_0^2(1 - \frac{T^2}{T_c^2})$

At the core of the string $\mid \phi \mid$ vanishes, it's thickness $\sim m_s^{-1}$. The magnetic field is determined by m_v^{-1},

$$m_v = e\eta = \frac{e}{\sqrt{\lambda}} m_s \qquad (14)$$

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[6]We should in principle sum over all the maxima of the functional, not just the string contributions. The approximation here has it's analogue in condensed matter with the independent vortex model of the λ transition in liquid Helium. There the model is in quantitative agreement with experiment, under the same assumptions we have made.

[7]These solutions are strictly speaking infinite string solutions. In dealing with loops of string, if we are to use these solutions, then we must work in the regime where the radius of curvature is large compared to the width of the string, so the string is approximately straight in any given region

and the string has an energy per unit length $\sigma = \sigma_s + \sigma_v$

$$\sigma_s = O(\eta^2(T)) \;\; scalar field$$
$$\sigma_v = O(\frac{e^2\eta^4}{m_v^4})$$
$$= O(\eta^2(T)) \;\; vector field$$

As $T \to T_c$, $\sigma \to 0$, $m_s^{-1} \to \infty$. The expression for I is substituted into Z, interactions between the strings are neglected by making them non- self intersecting, and placing them a distance l apart, where l is the lattice spacing which also corresponds to the width of a string. We then have:

$Z = \sum_n W(n)exp(-\beta\sigma ln)$

where $W(n)$ denotes the number of configurations of a string of length nl.

It is now possible to use standard results from polymer physics[20], to demonstrate the statistical properties of the strings at high and low string segment density[18]. Restricting ourselves to non self intersecting walks, at high string densities these are well approximated by Brownian walks, but not at low densities. We find the partition function for a 'gas' of loops is:

$$Z_{loop} = exp(Z_1)$$
$$= exp(\frac{CV}{2l^3} \sum_{n=1}^{\infty} n^{-q-1} e^{-\beta nl\sigma_{eff}})$$

where C is a normalisation constant, $q = \frac{3}{2}$ high string density, $\frac{7}{4}$ low string density. Z_1 is the partition function of the single loops, and the effective string tension is defined by:

$\sigma_{eff} = \sigma - \frac{In(a)}{\beta l}$

$a \sim 5$, depending upon the type of lattice the cosmic strings are placed on. We find that in the case of large n, both the contributions from infinite string (Z_∞), and closed loops (Z_{loop}) diverge for:

$$T > T_{st}$$
$$= \frac{\sigma l}{In(a)}$$

the temperature at which $\sigma_{eff} = 0$. Above this temperature there are large fluctuations in the ϕ field and it is no longer appropriate to describe the field in terms of string

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like configurations. In this region fluctuations in the energy with the canonical ensemble diverge (i.e the specific heat C_v diverges as $T \to T_{st}$), although the microcanonical ensemble remains perfectly well behaved above the Hagedorn temperature[17]. The reason for the breakdown in the energy is that there is an overcounting in the canonical ensemble for the number of possible string states becomes infinite. It is important to note however that this doesn't mean the canonical ensemble always breaks down. For example the fluctuation in the mean number of loops of length nl, is perfectly well defined all the way up to the Hagedorn temperature. In our case we find the fluctuations in the Higgs field are small enough just below the string temperature to make the canonical ensemble perfectly reasonable there. To see this, we evaluate C_v, as this gives us the fluctuations in the energy. When this is done, we find that the critical exponent is $\frac{1}{4}$ ($q = \frac{7}{4}$) and $\frac{1}{2}$ ($q = \frac{3}{2}$), implying the fluctuations are small, just below T_{st}. A very interesting result is obtained when we evaluate the partition function for an arbitrary number of dimensions d. We then have for the case of Brownian strings in d dimensions, $q = \frac{(d-1)}{2}$[17]. Now when we evaluate C_v, we find that the condition for it to remain finite in the limit $T \to T_{st}$ is that $d > 5$, (i.e for strings in 6 dimensions and above the fluctuations in the energy do not diverge in the canonical ensemble). Of course the actual value of T_{st} depends also on the number of dimensions.

We should think of T_{st} as the temperature at which our strings are formed,
$$T_{st} \simeq \sigma(T)l(T) \simeq \gamma \eta^2 m_v^{-1} \qquad \gamma \sim O(1) \qquad \lambda \gg e^2$$
Then it follows $T_{st} < T_c$ (i.e strings lower the phase transition temperature) with

$$\begin{aligned}
1 - \frac{T_{st}^2}{T_c^2} &= O(\lambda) \qquad , m = m_s \\
&= O(e^2) \qquad , m = m_v
\end{aligned}$$

The width of the strings at formation can now be calculated ($\sim$ mean separation)

$$\begin{aligned}
m_s(T_{st}) &= O(e m_s(T = 0)) \qquad , m = m_v \\
&= O(\sqrt{\lambda} m_s(T = 0)) \qquad , m = m_s
\end{aligned}$$

That is the network of strings at the phase transition has the separation of the centers of the flux tubes scaled up by a factor $O(\frac{1}{e^2})$, ($e^2 < \lambda$), a result obtained by Kibble[3], and discussed earlier in this lecture. This is a useful confirmation of the validity of the Kibble mechanism. What has become clear from the analysis is that as we

approach T_{st}, the mean field approach does not give the correct value for the critical temperature. We predict a second order phase transition at $T_{st} < T_c \mid_{meanfield}$. As the symmetry is restored, overlapping strings fill the whole of space. It is possible that thermal fluctuations cause a first order phase transition, indeed this has recently been invoked[31] in an attempt to investigate fundamental strings in the early universe. Above $T_{st} \sim O(T_{Ginzburg})$, our model is unable to recognise string like configurations. As T_{st} is approached the system becomes dominated by infinite string, with a scale invariant distribution of loops approximated by Brownian trajectories. Neglecting all interactions, actually implies Brownian trajectories, although it would be interesting to discover whether the simulations of[33] can detect this difference in Brownian versus non self intersecting random walks. Below T_{st} the system evolves to a state with an exponentially suppressed distribution of large loops.

We now move onto investigate the evolution of cosmic strings. For a string of width W and radius of curvature R, if $R \gg W$ then the action for a string is approximately a locally boosted version of the straight static solution. This is the Nambu action[21]:

$$S = -\mu \int d\sigma d\tau [-det g_{ab}^{(2)}]^{\frac{1}{2}} \tag{15}$$

where $g_{ab}^{(2)}$ is the world sheet metric,(a,b = 0,1). In terms of the string coordinates $X^\mu(\sigma, \tau)$ and the background spacetime metric:

$$g_{ab}^{(2)} = X_{,a}^\mu X_{,b}^\nu g_{\mu\nu}^{(4)} \tag{16}$$

The Nambu action (15) also has the interpretation of being the area of the two dimensional worldsheet. Corrections to the action and hence to the solutions to the equations of motion, are of $O(\frac{W}{R})$. Since for typical cosmic strings, $W \sim 10^{-30}m$, $R \sim kpc$ the corrections are small everywhere except near a cusp where $R \sim W$[22].

A vital issue for string theory concerns what happens when two strings collide? The Nambu action breaks down here, so the full non linear field equations must be solved (10). This was first studied numerically for the case of global strings by Shellard[23], who demonstrated that strings nearly always intercommute or exchange partners, for relative velocities below ·9. This is a very important result, as it is the only process by which loops can form in sufficient amounts from an initial configuration which contains nearly 80% of its length in infinite string. Analytical approaches to this problem, including introducing gauge fields have been tried, but only for special

cases[25]. One interesting prediction that Turok and myself came up with[25], which proved to be the case, was in showing that intercommuting always occurred when the strings first intersected. Whether or not they effectively reconnected later, depended upon the velocities of the new set of zeros that were produced in the interaction region. Basically four zeros would be produced after an initial intercommutation. These would then start moving out, two chasing behind the pair that had intercommuted and two in the direction the pair had before intercommutation. Depending on their velocities the original pair would be caught and annihilated leaving the remaining pair to carry on as if no intercommuting occurred. More often the ring of new zeros would not catch the original pair and would simply lose there energy as they spread out. The full analytical problem remains unsolved. Matzner has studied the case of local strings numerically and concluded they are even more likely to intercommute when they cross than the global case[24]. An open question remains to be resolved-what happens when two superconducting strings intercommute?

From (15), which is a purely geometrical object, we can write down the string equations of motion in say a flat Friedmann Robertson Walker universe (FRW). One important point to note is that the equations are independent of μ, the scale of the symmetry breaking. Reparameterisation invariance of S under $\sigma \rightarrow \overline{\sigma}(\sigma,\tau)$, $\tau \rightarrow \overline{\tau}(\sigma,\tau)$ enables a suitable gauge to be chosen:

$$x^0 = \tau \qquad\qquad \dot{\underline{x}}.\acute{\underline{x}} = 0 \qquad\qquad (17)$$

where $\dot{\underline{x}} \equiv \partial_\tau \underline{x}$, $\acute{\underline{x}} \equiv \partial_\sigma \underline{x}$. Then the equations of motion become, in terms of the scale factor $a(\eta)$ (where $dt^2 = a^2(\eta)d\eta^2$),

$$\ddot{\underline{x}} + 2\frac{\dot{a}}{a}(1 - \dot{\underline{x}}^2) = \frac{1}{\varepsilon}\frac{\partial}{\partial\sigma}(\frac{\acute{\underline{x}}}{\varepsilon}) \qquad\qquad \dot{\varepsilon} = -2\frac{\dot{a}}{a}\dot{\underline{x}}^2\varepsilon \qquad\qquad (18)$$

with

$$\varepsilon = [\frac{\acute{\underline{x}}^2}{1 - \dot{\underline{x}}^2}]^{(\frac{1}{2})} \qquad\qquad (19)$$

The energy in a string is

$$E = \mu a \int d\sigma\varepsilon \qquad\qquad (20)$$

In general it isn't possible to analytically solve these equations, but in certain regimes it can be done.

1]. If $R \ll$ damping term, then $\dot{\underline{x}} \sim 0$, and the string is simply conformally stretched with a radius of curvature $\gg H_0^{-1}$.

2]. If $R \ll \frac{a}{\dot{a}} = H_0^{-1}$, then the string doesn't notice the curvature of spacetime and acts as if it was in flat space. We can set $\varepsilon = 1$, and the equations become

$$\ddot{\underline{x}} = \underline{x}''$$

$$\dot{\underline{x}} \cdot \underline{x}' = 0 \tag{21}$$

$$\dot{\underline{x}}^2 + \underline{x}'^2 = 1$$

$$\tag{22}$$

The general solution is made up of right and left moving modes:

$$\underline{x}(\tau, \sigma) = \frac{1}{2}[\underline{a}(\sigma - \tau) + \underline{b}(\sigma + \tau)] \tag{23}$$

with the gauge condition implying:

$$\underline{a}'^2 = \underline{b}'^2 = 1$$

in the centre of mass frame of the string. Thus $\underline{a}'$, $\underline{b}'$ are closed curves on the unit sphere, generally intersecting if they are continuous[26]. In fact the picture for $R \ll t$, is that loops of string break off the network of long string and oscillate as if in flat space. At an intersection $\underline{x}' = 0$, $\dot{\underline{x}}^2 = 1$ and such a point is a cusp (defined by the vanishing of $\det g_{ab}^2$). However as $\underline{x}'$ need not be continuous, such points where it isn't are called kinks. Four kinks are always produced when a string intersects.

We can summarise the early string evolution picture. Initially 80% is in infinite string. As the universe expands, the infinite strings chop off loops, (which sometimes reconnect back onto an infinite string). These loops also self intersect until a class of non-self intersecting loop solutions are reached. In general these loops are stable against perturbations which could cause them to self intersect, (i.e from say matter accreting around such a loop)[27]. Eventually these loops will be identified with the galaxies and clusters of galaxies. Many problems remain unresolved. What is the region of phase space which possess non-self intersecting loops? It has been shown to be non zero, but if we want to predict the number of daughter loops an initial parent loop will produce, we need to be able to answer that question. Progress

towards answering this issue has recently been made in that it is now possible, by studying just the kinks and cusps present on a loop to place a lower bound on the total number of times a given parent loop will self intersect[28]. How generic are cusps on strings, and how do we deal with kinks are two more open questions? We still need to know how the gravitational backreaction from an oscillating loop affects its' motion. Does it cause the loop to lose its' periodicity? As yet no exact analytical solution for a cosmic string in an expanding universe has been found, so there are plenty of important analytical issues still to be resolved.

As the loops oscillate, due to their tension they decay primarily by emitting gravitational radiation[29]. Using the quadrupole formula, it is possible to estimate the lifetime of the loop at about 10^6 oscillations. Currently work is in progress to accurately deal with the backreaction on a loop due to it's radiation.

The 'loop production function' is an important measurement device used in the evolution scenario. If the rate of loop formation from infinite strings is too small, then the strings quickly dominate the total energy of the universe. As we have seen, energy in strings longer than H_0^{-1}, (several per horizon volume), scales as a(t), whereas energy in radiation scales as $a^{-1}(t)$. Thus $\frac{\rho_\infty}{\rho_{rad}} \propto a^2(t) = t$. These strings must chop off a constant fraction of their length each expansion time into loops to avoid this problem. The way this works is that the correlation length, $\xi(t)$ of an infinite string at time t, is t. This is the condition for the scaling solution which is vital if strings are not to dominate the universe. To see this we know at formation

$$\rho_{string} = \frac{\mu\xi}{\xi^3} = \frac{\mu}{\xi^2}.$$

Thus

$$\frac{\rho_{string}}{\rho_{rad}} = const \ if \ \xi \propto t.$$

In a scaling solution the only length scale is H_0^{-1}. If events occurring 10^{-35} seconds after the big bang are to explain events some 10 billion years later, we need some sort of scaling if we are to be able to predict anything. Fortunately the numerical results do show scaling[30], and analytically we can explain why small loops would form rather than infinite string. It is due to the increased amount of phase space available in flat space for loops[17,18].

Another important issue which will hopefully soon be resolved in the numerical simulations is the role of the smallest loops that are produced. Loops decay slowly, so their density scales as matter, decreasing slower than in the long strings. This means that small loops dominate the energy density in strings. Once again the importance of the chopping process terminating after a finite number of intersections is demonstrated. We now have a distribution of loops with a scaling solution, the only length scale is H_0^{-1} and mass scale is μ. What does this model have to offer with regard to the formation of structure, how does matter accrete around these loops and how efficiently? In the next lecture I will address these issues.

2 Lecture 2: Large Scale Structure

Most of the mass in the universe is dark, the evidence for this comes from many quarters:

1]. The fact that the spiral galaxy rotation curves stay flat much further than their visible radius implies $\Omega_{halos} \sim \Omega_{clusters} \sim \cdot 1 - \cdot 3$

2]. Nucleosynthesis sets bounds on the baryonic matter [34], $\cdot 04 < \Omega_B h_{50}^2 < \cdot 14$, yet $\Omega_{luminous} < \cdot 02$. Thus most of the baryons are dark.

3]. Inflation predicts Ω should be close to one.

Fortunately particle physics provides us with many potentially exotic sources for the dark matter, these could be hot, (i.e neutrinos), cold (i.e axions), and they could be supersymmetric[35].

Now the hot big bang is incomplete without a source of perturbations which were essential to seed structures on large scales, especially as the universe was essentially very isotropic and homogeneous early on. Initially we will investigate the density perturbations due to an infinite straight string[36]. Recall the action (15) [21]:

$$S = -\mu \int d\tau d\sigma [-det g_{ab}^2]^{\frac{1}{2}} \tag{24}$$

we can write the stress-tensor:

$$
\begin{aligned}
T^{\mu\nu}(x) &= -\frac{2}{\sqrt{-g^4(x)}} \frac{\delta S}{\delta g_{\mu\nu}(x)} \Big|_{g=\eta} \\
&= \mu \int d\sigma \begin{pmatrix} 1 & \dot{x}^i \\ \dot{x}^i & \dot{x}^i \dot{x}^j - \acute{x}^i \acute{x}^j \end{pmatrix} \delta^3(x - x(\sigma,\tau))
\end{aligned} \tag{25}
$$

For a straight static string along the z-axis, $\underline{x}(\sigma,\tau) = (0,0,\sigma)$, so:

$$T^{\mu\nu}(x) = \mu\delta^2(x)diag(1,0,0,-1) \tag{26}$$

Note the role of the negative pressure term, this gives the string its' tension and will prove important in the superconducting cosmic strings scenario[37]. What is the effect of this string on the background metric? We write,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x),$$

which is valid provided $\mid h \mid \ll 1$. In the Harmonic gauge, $(g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0)$, Einsteins' equations become

$$\partial^2 h_{\mu\nu} = -16\pi G(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\lambda_\lambda) \tag{27}$$

The time independent solutions have no Newtonian potential for h_{00}, h_{33}, (i.e $\nabla^2 h_{00} = \nabla^2 h_{33} = 0$). However we do obtain

$$h_{11} = h_{22} = 8G\mu In(\frac{r}{r_0}) \tag{28}$$

where r_0 is a suitable cut off introduced into the integral. Thus we find the line element becomes:

$$ds^2 = dt^2 - dz^2 - (1 - 8G\mu In(\frac{r}{r_0}))(dr^2 + r^2 d\phi^2 \tag{29}$$

Redefining
$d\acute{r} = (1 - 4G\mu In(\frac{r}{r_0}))dr$
$d\acute{\phi} = (1 - 4G\mu)d\phi$
to obtain

$$ds^2 = dt^2 - dz^2 - (d\acute{r}^2 + \acute{r}^2 d\acute{\phi}^2) \tag{30}$$

we see that the effect of the backreaction of the string on the spacetime is to leave the metric flat everywhere except at the vertex, with an accompanying missing angle, $0 < \acute{\phi} < 2\pi - 8\pi G\mu$[36]. The spacetime is conical shaped with the string passing through the vertex of the cone. However the gravitational attraction due to the string on a static test particle is zero, as the particle effectively experiences a flat spacetime. If the string is moving relative to the test particles, they will feel the conical spacetime and be drawn in behind the string over an angle $2\Delta\phi = 8\pi G\mu$, forming small wakes where $\rho_{wake} = 2\rho_{background}$ as matter falls in on either side of the

wake. I will mention observational consequences of this missing angle at the end of the lecture.

We saw in the previous lecture how most string ends up in loops. By the time matter starts accreting around these loops they should be well inside the horizon, $r \ll H_0^{-1}$, so gravity should then be weak inducing small peculiar velocities, $\delta v \ll 1$. In this regime we can use linear perturbation theory[38]. Also on scales large compared to their size, it is a good approximation to treat the loops as Newtonian point particles[39]. Thus in the Newtonian approximation to Einsteins equations we have for a particle at $\underline{r}(t)$, (for more details see[9]):

$$\ddot{\underline{r}} = -\underline{\nabla}_r \Phi \tag{31}$$

where the Newtonian potential obeys

$$\underline{\nabla}^2 \Phi = -4\pi G \rho \tag{32}$$

For cold dark matter, (CDM), mass is conserved, so in a comoving volume

$$\int \rho dV_{comoving} = const \tag{33}$$

The solution in a homogeneous background, $\rho_B(t)$ only is

$$\begin{aligned}
\Phi &= \frac{4\pi G \rho_B \underline{r}^2}{6} \\
\ddot{\underline{r}} &= -\frac{4 G \rho_B \underline{r}}{3} \\
\rho_B r^3 &= const
\end{aligned} \tag{34}$$

In an $\Omega = 1$ universe, we find on integrating (34),

$$\begin{aligned}
r &\propto t^{\frac{2}{3}} \\
\rho_B &= \frac{1}{6\pi G t^2}
\end{aligned} \tag{35}$$

which is the standard FRW matter dominated universe. Now we want to do linear perturbation theory about this solution. Define $a(t) = (\frac{t}{t_i})^{\frac{2}{3}}$ where t_i is the initial time. Then

$$r(t) = a(t) r_i \tag{36}$$

Thus $\underline{x} = \underline{r}_i$ is the initial comoving coordinate of the particle. Giving each particle a small comoving displacement $\underline{\psi}(x,t)$ we write

18

$$\underline{r}(t) = a(t)[\underline{x} + \underline{\psi}(\underline{x}, t)] \tag{37}$$

Mass conservation implies,

$$\rho_i d^3 x_i = \rho(r) d^3 r \tag{38}$$

so using $\rho_i = \rho_B a^3$, this implies

$$\rho(\underline{r}) = \frac{a^3 \rho_B}{|\frac{dr}{dx}|}$$

From (37) we have:

$$\rho(\underline{r}) = \rho_B (1 - \underline{\nabla}_x \cdot \psi) + O(\psi^2) \tag{39}$$

Defining

$$\frac{\delta \rho}{\rho} \equiv \frac{(\rho(r) - \rho_B)}{\rho_B}$$

we can write

$$= -\underline{\nabla}_x \cdot \underline{\psi} \tag{40}$$

for the fractional density perturbation.

To solve for r(t), hence for $\psi(\underline{x}, t)$, substitute (39) into (31,32)

$$\begin{aligned} \nabla_r^2 \Phi &= \frac{1}{r^2} \frac{d}{dr} (r^2 \frac{d\Phi}{dr}) \\ &= -4\pi G \rho(r) \end{aligned}$$

with

$$(\underline{\nabla}_r = \frac{\underline{\nabla}_x}{a})$$

and obtain

$$r^2 \frac{d\Phi}{dr} = -4\pi G \int r^2 dr \rho_B (1 - a\underline{\nabla}_r \cdot \underline{\psi}) \tag{41}$$

Hence we find

$$\underline{\nabla}_r \Phi = 4\pi G \rho_B (\frac{r}{3} - a\underline{\psi})$$

which when substituted in (31) gives

$$\ddot{\underline{\psi}} + \frac{2\dot{a}}{a} \dot{\underline{\psi}} = 4\pi G \rho_B \underline{\psi} \tag{42}$$

Taking the homogeneous solution we find $\dot{\underline{\psi}} \propto a^{-2}$. Now the peculiar velocity of a test particle (the physical velocity minus the Hubble flow) is,

$$
\begin{aligned}
v_p &= \dot{\underline{r}} - \frac{\dot{a}}{a}\underline{r} \\
&= \dot{\underline{r}} - \dot{a}(\underline{x} + \underline{\psi}) \\
&= a\dot{\underline{\psi}}
\end{aligned}
$$

or

$$
v_p \propto \frac{1}{a},
$$

so a test particle slows down in comoving coordinates. We interpret the right hand side of (42) as the gravitational instability, the feedback of the perturbation on itself. For the case of the matter dominated FRW universe, i.e $\rho_B = \frac{1}{6\pi G t^2}$, the solution to (42) is

$$
\underline{\psi}(x,t) = \underline{A}(x)t^{\frac{2}{3}} + \underline{B}(x)t^{-1} \tag{43}
$$

with $\underline{A}$ and $\underline{B}$ arbitrary functions. Recalling (40) we see that $\frac{\delta\rho}{\rho}$ obeys the same equation as ψ, it has the same time behaviour. Thus

$$
\frac{\delta\rho}{\rho} \propto a(t)
$$

in the growing mode.

What happens in the case when there is a local non-linear mass concentration? Is it still correct to use the above equations which are only valid in the linear regime? The answer turns out to be yes, provided the density perturbation is small on the surface surrounding the mass concentration. To see this we follow an argument presented in [9]. Surrounding a volume V is a comoving surface S. The average density in a comoving volume (for CDM) is, using (38)

$$
\begin{aligned}
\overline{\rho}V_f &= \rho_i V_i \\
&= \rho_B a^3 V_i
\end{aligned} \tag{44}
$$

where $\overline{\rho}$ is the average density, V_f, V_i are the final and initial volumes. Now we can write $V_f = a^3(V - \delta V)$ where

$$
\delta V = \int_S \underline{\psi}.d\underline{S} \tag{45}
$$

Writing

$$\bar{\rho} = \rho_B \frac{a^3 V_i}{V_f} \simeq \rho_B (1 + \frac{\int_S \underline{\psi}.d\underline{S}}{V_i}) + O(\psi^2)$$

we easily see

$$\frac{\delta \rho}{\rho} = \frac{\int_S \underline{\psi}.d\underline{S}}{V_i}$$

$$\propto t^{\frac{2}{3}} \; in \; the \; growing \; mode \tag{46}$$

In order to apply the linear regime to mass concentrations, we need linearity on S, not in V.

Turning our attention back to peculiar velocities, recall

$$v_p = \underline{\dot{r}} - (\frac{\dot{a}}{a})r = a\underline{\dot{\psi}}.$$

For the solution (43), in the growing mode, $v_p \propto t^{\frac{1}{3}}$. Now (43) also tells us that in the growing mode, $\psi \propto a$, whereas from (37) we know

$$\underline{\psi} = \frac{(\underline{r} - \underline{x})}{a}.$$

Hence

$$\underline{v}_p = (\frac{\dot{a}}{a})[\underline{r} - \underline{x}] = Hd\underline{r} \tag{47}$$

the equation used in lecture 1. The usefulness of (47) is that in particular geometries v_p is related to the magnitude of $\frac{\delta\rho}{\rho}$. Defining the Hubble velocity, $v_H = Hr$, write

$$\frac{v_p}{v_H} = \frac{d\underline{r}}{r} \tag{48}$$

In CDM scenarios we know mass is conserved in a comoving volume (33). Equation (46) tells us that

$$\frac{\delta \rho}{\rho} = \frac{\int \underline{\psi}.d\underline{S}}{V_i}.$$

For planar collapse over a length scale L, $V_H = HL$, so

$$\frac{\delta \rho}{\rho} = \frac{\psi}{L} = \frac{v_p}{V_H}.$$

For cylindrical collapse it is easy to see, (because $V \propto r^2$)

$$\frac{v_p}{v_H} = \frac{\delta\rho}{2\rho},$$

and for spherical collapse, $(V \propto r^3)$,

$$\frac{v_p}{v_H} = \frac{\delta\rho}{3\rho}.$$

Recent observations of peculiar velocities give a unique window on $\frac{\delta\rho}{\rho}$[40]. In particular for the 'great attractor', $v_p \sim 500 kms^{-1}$, at $L \sim 100 h_{50}^{-1} Mpc$ gives $\frac{\delta\rho}{\rho} \sim \cdot 3$.

So far we have thought only about linear collapse in a particular background. What happens when we include sources, (e.g loops of cosmic string)[9]? On scales large compared to the size of the loop, we can treat the loops as Newtonian point particles. Equations (32,42) become

$$\nabla^2 \Phi = -4\pi G(\rho_B + \rho_{source}) \tag{49}$$

$$\ddot{\underline{\psi}} + \frac{2\dot{a}}{a}\dot{\underline{\psi}} - 4\pi G\rho_B \underline{\psi} = -\frac{1}{a}\nabla\Phi_{source} \tag{50}$$

A nice trick used to solve (50) is found by recalling that we really require $\frac{\delta\rho}{\rho}$. Using Gauss's theorem to rewrite (46)

$$\frac{\delta\rho}{\rho} = \frac{\int \nabla.\underline{\psi}dV}{V_i} \tag{51}$$

we define $\delta \equiv \nabla.\underline{\psi}$, and take the divergence of (50)

$$\ddot{\delta} + \frac{4}{3t^2}\dot{\delta} - \frac{2}{3t^2}\delta = 4\pi G\delta\rho_{source}(\underline{x},t) \tag{52}$$

This has a Green's function solution,

$$\delta(\underline{x},t) = \int_{t_i}^{t} dt' G(t,t') 4\pi \delta\rho_{source}(\underline{x},t)$$

where

$$G(t,t') = \frac{3}{5}(t^{\frac{2}{3}}(t')^{\frac{1}{3}} - t^{-1}(t')^2).$$

Now for a loop with comoving trajectory $Z(t,\sigma)$,

$$\delta\rho_{source}(t) = \mu \int d\sigma \frac{\delta^3(\underline{x} - \underline{z}(t,\sigma))}{a^3(t)}$$

which leads to the trumpet shape trajectories as the matter falls onto a loop and the loop shrinks in comoving coordinates[41].

We move on to describe briefly the case of non - linear collapse. More complete treatments can be found in [9], or in Brandenbergers review article[2]. Non-linear collapse can only be solved for special geometries. The spherical collapse model is ideal for the case $r \gg R_{loop}$, where R_{loop} is the radius of the loop and r is the distance from the loop. Then the time averaged potential from the loop is
$\Phi = -\frac{Gm}{r}$, where m is the mass of the loop. For $v \ll 1$, the Newtonian approximation is good. In a matter dominated universe, the pressure vanishes, so one solves the equation for r(t). Defining $M = m + M_i$, where M_i is the mass inside a shell of radius r_i,

$$M_i = \frac{4}{3}\pi \rho_i r_i^3 = \frac{2r_i^3}{9Gt_i^2}$$

in a matter dominated universe, we find in this approximation

$$\frac{\dot{r}^2}{2} - \frac{GM}{r} = const \quad < \quad 0 \quad bound\ solution$$
$$> \quad 0 \quad escape$$

For an $\Omega = 1$ universe, then, const $= 0$, all the shells are bound to the loop. Imposing the initial condition of unperturbed Hubble flow, $\dot{r}_i - H_i r_i = 0$, the parametric solution for small $\delta = \frac{m}{M}$ is

$$r = \frac{r_i(1 - cos\theta)}{2\delta_i} \qquad\qquad t = \frac{3t_i(\theta - sin\theta)}{4\delta_i^{\frac{4}{3}}} \qquad (53)$$

In the early stages of collapse,

$$\frac{\delta\rho}{\rho} = \left(\frac{r_i(\frac{t}{t_i})^{\frac{2}{3}}}{r}\right)^3$$

from the simple scaling of the universe as it expands. Also from $M \propto r^3$,

$$\frac{\delta\rho}{\rho} \propto \delta_i \left(\frac{t}{t_i}\right)^{\frac{2}{3}}.$$

The radius r turns round when $\theta = \pi$, $(\dot{r} = 0)$, with $\frac{\delta\rho}{\rho_B} = (\frac{3\pi}{4})^2 - 1 = 4 \cdot 5$. The shell initially moves outwards with the expansion of the universe, slows down under gravity and eventually collapses after reaching a maximum radius. Roughly

speaking at turnaround, all the energy is potential energy, $P.E = -\frac{GM}{r_{max}}$. The shell recollapses eventually virialising (i.e kinetic energy $= \frac{1}{2}P.E_{max}$). Therefore at virialisation, $r_{virial} = \frac{r_{max}}{2}$. Knowing this we can write

$$\frac{\delta\rho}{\rho} \mid_{vir} = 8.(4\cdot5).(\frac{a_{vir}}{a_{max}})^3 \tag{54}$$

The factor of 8 comes from the collapse to $\frac{r_{max}}{2}$, and the scale factor dependence is from $\rho_B \propto a^{-3}$. From (54) we can estimate $\frac{\delta\rho}{\rho} \mid_{vir}$ for Abell clusters and galaxies. We find for the case of CDM

$$\frac{\delta\rho}{\rho} \mid_{vir} \simeq 150 \ with \ 1 + Z_t \ \sim \ 1\cdot5 \ (Abell \ Clusters)$$

$$\frac{\delta\rho}{\rho} \mid_{vir} \simeq 10^{4-5} \ with \ 1 + Z_t \ \sim \ 6 - 13 \ (Galaxies)$$

thus we now have an observational test for the theory(see Turok and Brandenberger in[39]). Fitting todays values for the observed overdensities in Abell clusters, $\frac{\delta M}{M} \sim 170\Omega^{-1}$, with, $\Omega = 1, h = \cdot5$, (i.e Abell clusters have only recently virialised), we find that for cosmic strings to be consistent, it implies $G\mu \simeq 2.10^{-6\pm1}$. This fits very well with the prediction of GUT scale symmetry breaking, ($\mu \sim 10^{16}Gev$), yet is independent of that result. With this value there are now no free parameters in the theory, and it is possible to show how strings predict the correct masses of the galaxies(see Turok and Brandenberger in [39]). The case of Hot Dark Matter is different in the details of the calculation, and I advise interested readers to refer to [9], and Brandenbergers article[2] for details.

Finally in this lecture I will discuss some of the observational tests of cosmic string theory, tests on which the model will probably either succeed or fail. They all require a determination of $G\mu$. The first test I have already mentioned is the gravitational lensing by strings[36], where an observer could see a double image of a quasar or galaxy behind a cosmic string. The images would be separated by an angle of $4\pi G\mu \sim 5'''$, (the missing angle), for $G\mu \sim 10^{-6}$. There are five or so reported cases of lensing between $2.5''' - 7'''$. However a line of double images would be strong evidence for the existence of a string.

Detection of anisotropy in the local microwave background radiation could be evidence for cosmic strings, as first noted by Kaiser and Stebbins[15]. They realised that strings produced a background temperature with steplike discontinuities on curves in

the sky. As with the lensing, this relies on the canonical structure of spacetime near a string. Light rays travelling on either side of the string produce a Doppler shift, $\Delta\nu = 8\pi G\mu v_\perp$ in their frequency, $v_\perp$ is the transverse velocity of the string. Co-moving frames on either side of the string move towards each other with a velocity, $v_\perp \sim \cdot4$. Thus we find

$$\frac{\delta T}{T} = 8\pi G\mu v_\perp$$
$$< 2.10^{-5}$$

which is almost at the level of accuracy we can now observe. Recently temperature maps of a 'stringy' universe have been produced to show typical angular distributions of the discontinuities, hence what the temperature distribution may typically look like[42].

Possibly the tightest constraint on cosmic strings comes from the gravitational radiation from loops of string[43,29,15]. Typically the energy density of radiation from a loop, in terms of the logarithmic spectrum,

$$\Omega_g(w) = w\rho(w)\rho_0^{-1},$$

($w = R^{-1}$ is the main frequency of radiation, ρ_0 is the background energy density), is

$$\Omega_g(w) = 2.10^{-7}h_{50}^{-2}.$$

Now variations in the observed frequency of the millisecond pulsar, places the severest restriction on $G\mu$,

$$G\mu \leq 10^{-5}(\frac{\alpha}{T_{yr}})^8$$

where $\alpha \sim 2\pi$, T_{yr} is the observation period in years. In principle we will soon be in a position to confirm or rule out cosmic strings. However in practice we are reaching the limits of reliability with the pulsar timing, both in our knowledge of the dynamics and content of the solar system and the accuracy of the atomic clocks we use to standardise things with. Both are reaching the limits beyond which we wouldn't trust conclusions inferred from their results. The latter problem could be overcome by using a few pulsars and comparing their relative times, as they are the most accurate clocks we know of. For the time being strings are still very much alive.

The agreement Turok found between the correlation of Abell size loops and the observation of the Abell clusters[44], created a great deal of interest in strings. Today more observations are testing the theory. Will the Abell result remain both in the simulations and the observations? What will the higher correlation functions look like. Currently a great deal of observation time is going into investigating the large scale structure of the universe, including things like 3pt correlation functions between clusters, the distribution of bubbles, voids, filaments. The next few years will be an exciting time in comparing the numerical simulations with observation. Work is currently going on to investigate the growth of matter around a network of string, at least in the linear regime. The full N-body problem is also being investigated. One interesting result from strings is that it predicts correlation functions should be universal, because of the scaling solution for strings.

Superconducting cosmic strings also produce possible correlations between clusters of galaxies[5]. They could provide the most explosive evidence for strings, as we shall see in the next lecture!

3 Lecture 3. Superconducting Cosmic Strings

In 1984, Witten[4] realised that in some models, ordinary cosmic strings could carry very large currents along them, behaving like superconducting wires. The charge carriers accounting for the current, could either be bosonic, where a charged Higgs field has an expectation value in the core of the string, or fermionic where charged fermions are trapped as zero modes along the string.

In this lecture I will concentrate mainly on the case of bosonic superconducting cosmic strings, although I will discuss some of the results of the fermionic case. Several astrophysical consequences of the superconducting cosmic strings have been suggested[4,5,45,46,47]. We will review these consequences, explain how to obtain and understand the motion of[37,48], and radiation from[49,48,50] superconducting cosmic strings. What will become apparent is the importance of the current to the equations of motion, and the fact that the total radiation from the loops is finite[50] contrary to previous estimates.

First of all, let us see how superconducting cosmic strings arise[4]. In lecture 1, we investigated the complete breaking of an Abelian group $G=U'(1)$ with lagrangian

(10)

$$\mathcal{L}[\phi, A_\mu] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} \mid D_\mu\phi \mid^2 -\frac{\lambda}{4!}(\mid \phi \mid^2 -\eta^2)^2 \tag{55}$$

Instead of this $U'(1)$ theory, we consider a $U'(1) \times U(1)_{em}$ where the $U'(1)$ symmetry is broken in the true vacuum producing strings as before, but the ϕ field is now coupled to an electrically charged χ field:

$$\mathcal{L}[\phi, R_\mu, \chi, A_\mu] = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}R_{\mu\nu}^2 + \mid D_\mu\phi \mid^2 + \mid D_\mu\chi \mid^2 -V(\phi,\chi) \tag{56}$$

where

$$V(\phi,\chi) = \frac{\lambda_1}{4}(\mid \phi \mid^2 -\eta^2)^2 + \frac{\lambda_2}{2} \mid \chi \mid^4 +\lambda_3(\mid \phi \mid^2 -m^2) \mid \chi \mid^2 \tag{57}$$

with $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}, R_{\mu\nu} = \partial_{[\mu}R_{\nu]}$
$D_\mu\chi = (\partial_\mu + ieA_\mu)\chi, D_\mu\phi = (\partial_\mu + igR_\mu)\phi$
Initially $U(1)_{em}$ is unbroken, whereas $U'(1)$ is broken producing the usual cosmic strings. To ensure electromagnetism is unbroken outside the string,

$$\frac{m^2}{\eta^2} \leq 1 \quad \frac{\lambda_1\lambda_2}{2\lambda_3^2} > \frac{m^4}{\eta^4}.$$

The potential is then minimised with $< \chi >= 0$, (electromagnetism unbroken) and $\mid< \phi >\mid= \eta$ ($U'(1)$ spontaneously broken) in the vacuum. However as we know from figure 1, $< \phi >= 0$ in the core of the string. The potential (57) for χ then has the symmetry breaking form and forces χ to be non zero in the core. In fact we find $< \chi >\sim \sqrt{\frac{\lambda_1}{\lambda_2}}m$ in the core. It is easy to demonstrate that for some range of the couplings, a χ condensate does exist on the string[4,52,51,53,54]. We will return to this point later.

Our solutions are constructed around an arbitrary curved worldsheet with spacetime coordinates $x^\mu(\sigma^a)$, where $\sigma^a = (\tau,\sigma)$ are the two worldsheet coordinates. Given such a worldsheet we construct two spacelike normal vectors, $n_A^\mu(\sigma)$ ($A = 1,2$) which everywhere obeys $n_A^\mu x_{\mu,a} = 0$. We choose them to be orthonormal ($n_A^\mu n_{\mu B} = -\delta_{AB}$). For any point y in spacetime closer to the string than it's radius of curvature, we can associate two worldsheet coordinates σ^a and two radial coordinates ρ^A:

$$y^\mu = x^\mu(\sigma) + n_A^\mu(\sigma)\rho^A$$

$\chi = \chi_0(\rho)$ in the condensate. Now if $\chi_0(\rho)$ minimises the energy of the string, so does $e^{i\theta}\chi_0(\rho)$ for any real constant phase θ, although χ then carries no current. What Witten noticed was that at little energetic cost, the string had low energy excitations of the form:

$$\chi(\rho, z, t) = e^{i\theta(z,t)}\chi_0(\rho) \tag{58}$$

which led to persistent currents in the string. Substituting (58) into the static straight string solution along the z- axis, there is an extra contribution to the string action (56):

$$\begin{aligned} \Delta S_\chi &= \int d^2\sigma d^2\rho\sqrt{-\gamma} \mid D_a\chi \mid^2 \\ &= K \int d^2\sigma\sqrt{-\gamma}\gamma^{ab}(\partial_a\theta + ieA_a)(\partial_b\theta + ieA_b) \end{aligned} \tag{59}$$

where $K = \int d^2\rho \mid \chi_0^2(\rho) \mid^2$. The term $\mid D^4\chi_0(\rho) \mid^2$ is included in the definition of μ, (compare (11)), and to $O(\rho)$ we are left with the $\mid D_a\chi \mid^2$ term in (59). Assuming the gauge field A_μ varies slowly across the string, we have used

$$A_a(\sigma) = x^\mu_{,a}A_\mu(x(\sigma))$$

Since

$$\chi_0 \sim \sqrt{\frac{\lambda_3}{\lambda_2}}m$$

in the core of the string, and the width of the χ condensate

$$\sim \frac{1}{\sqrt{\lambda_3 m^2}}$$

from (57), we estimate $K \sim \frac{1}{\lambda_2}$.

The electromagnetic current

$$J_a \equiv -\frac{\delta(\Delta S_\chi)}{\delta A^a} = -2eK(\partial_a\theta + eA_a) \tag{60}$$

enables us to rewrite (59),

$$\Delta S_\chi = \frac{1}{4e^2 K}\int d^2\sigma\sqrt{-\gamma}\gamma^{ab}J_aJ_b \tag{61}$$

Now the superconducting property of the string arises because the total change in θ around the loop is conserved as long as $|\chi|$ does not go to zero, (i.e does not hop over the potential barrier) in the core of the string. The topologically invariant quantity is the winding number N,

$$N = \frac{1}{2\pi} \oint dl \frac{\partial \theta}{\partial l} \tag{62}$$

where l is a parameter along the string. Despite being the integral around a closed loop of the derivative of θ, N need not vanish, as θ is defined moduli 2π. N can be any integer. Thus from (60,62), if a situation is set up where $N \neq 0$, then $J_1 \neq 0$ corresponds to the ground state of the string carrying current. The winding of θ can change through the quantum tunneling of the χ field, so the current carrying state is metastable[4,52,51,53,54]. However for natural values of the coupling constants this is exponentially suppressed.

Maxwells equations enable us to determine the gauge field A_a in terms of $\partial_a \theta$. In the Lorentz gauge, $\partial_\mu A^\mu = 0$, these read[4,50],

$$
\begin{aligned}
\partial^2 A^\mu(y) &= \int d^2\sigma \sqrt{-\gamma} \delta^4(x(\sigma) - y) J^a \partial_a x^\mu \\
&\equiv J^\mu(y)
\end{aligned}
\tag{63}
$$

This has a solution in terms of the retarded Greens function,

$$G^{ret}(y) = \frac{1}{2\pi} \delta(y^2) \theta(y^0)$$

We find that for a loop with radius of curvature R, string width W, in the regime $\frac{R}{W} \gg 1$, (i.e typically),

$$A^\mu(x(\sigma)) \simeq \frac{1}{2\pi} In(\frac{R}{W}) J^a \partial_a x^\mu(\sigma) \tag{64}$$

as the leading term. The next order terms are down by $O(In(\frac{R}{W}))^{-1}$. Now for a wire of width W, carrying a uniformly distributed current J, (64) tells us the gauge field interior to the surface $A_I = A_0 + \frac{J}{2\pi}(\frac{\rho}{W})^2$, where A_0 is the value of the field at the centre of the string. So the variation of A across the string is $O(\frac{\rho^2}{W^2})$ which is negligible as previously assumed. Putting (64) into (60) we find that on the string

$$
\begin{aligned}
J_a &= 2eK_{eff} \partial_a \theta \\
A_a &= -(\frac{K_{eff}}{\pi}) \partial_a \theta \\
K_{eff} &= \frac{K}{(1 + \frac{Ke^2}{\pi} In(\frac{R}{W}))}
\end{aligned}
\tag{65}
$$

formula originally derived in [4] for a straight string. Typically $K_{eff} \sim \cdot 25$, (needed for loops to last a cosmologically significant length of time),with $a_{em} \sim \frac{1}{137}$, $In(\frac{R}{W}) \sim 100$.

This still isn't the full story to the correction to the action. So far we have obtained ΔS_χ, and J_a, A_a in terms of $\partial_a \theta$. Equation (56) reminds us of the electromagnetic contribution of the superconducting cosmic string action.

$$
\begin{aligned}
S_{em} &= -\frac{1}{4} \int d^4 y F_{\mu\nu} F^{\mu\nu} \\
&= \frac{1}{2} \int d^4 y A^\mu \partial^2 A_\mu \quad \text{in the Lorentz gauge.}
\end{aligned}
\tag{66}
$$

As previously we solve Maxwells' equations for A_μ in terms of J_μ, hence J_a to obtain[50]

$$
\Delta S_{em} = \frac{1}{4\pi} In(\frac{R}{W}) \int d^2\sigma \sqrt{-\gamma} \gamma^{ab} J_a J_b
\tag{67}
$$

as the leading local contribution to the string action. Adding (67) and (61) and rescaling the current, we find that the action for superconducting cosmic strings in the absence of external magnetic fields to be[37]

$$
S = -\mu \int d^2\sigma \sqrt{-\gamma} [1 - \gamma^{ab} j_a j_b]
\tag{68}
$$

$$
j_a = \frac{J_a}{J_S}
$$

Here j_a is a dimensionless string current. J_S will prove to be the string saturation current. Under what conditions is (68) valid? Certainly not in regions where $R \sim W$, i.e kinks or cusps. It ignores non- local self interactions of the string with itself, by radiation from one part affecting another part of the string. Actually these affects are down by a factor $(In(\frac{R}{W}))^{-1} \sim \cdot 02$. For the case of the infinite strings, we have dropped surface terms in S_{em} which makes (68) invalid[55,53]. The affect is to add extra terms to the stress tensor. We also assumed the existence of the current on the string did not affect $\chi_0(\rho)$ which was taken to be independent of J. This approximation is good in regions of parameter space where the Higgs coupling in (57) takes natural values, (i.e not small relative to e^2). Outside these regions, one must investigate the contribution case by case. We can safely say that (68) is valid to all orders in natural regions of parameter space. It agrees to $O(j^2)$ with that based on a Kaluza- Klein construction[56], however their action is valid only for small j.

Before discussing solutions to (68), we will review some of the proposed astrophysical consequences of superconducting cosmic strings. Chudnovsky et al.[45] considered their affect on plasma, estimating the synchrotron emission and radiation from shock heating. Massive strings with $G\mu \sim 10^{-6}$, have been invoked by Ostriker, Thompson and Witten, [OTW][5], as the power sources for giant explosions. We will look more closely at their very attractive scenario here. The recent survey of the northern sky[8] seems to indicate that galaxies are found on the surfaces of bubbles of the size$\sim 20h_{50}^{-1}Mpc$, the surfaces surrounding voids which are apparently empty of galaxies. So far there has been no real explanations from ordinary cosmic strings for the existence of these bubbles or voids, (I should point out here, as more data is being taken, some of the voids are beginning to close in, proving to be not as empty as was first thought). However superconducting cosmic strings do seem to offer a natural explanation for these voids. We know that as loops oscillate, gravitational waves are radiated with a luminosity, $L_g = \gamma_g G\mu^2$, causing the loop to shrink, ($\gamma_g \sim 50 - 100$ independent of the loop size). Now because superconducting cosmic strings carry an electrical current, $J < J_{max} \sim e\sqrt{\mu}$, an oscillating loop will also emit electromagnetic radiation. We can generally write the electromagnetic luminosity, $L_{em} = \gamma_{em}j^2\mu$, $where$ $j \equiv \frac{J}{J_{max}}$, $\gamma_{em} \sim 10$, again independent of size. Later we will see how the string equations of motion modify the power radiated. OTW show that for $G\mu \sim 10^{-6}$, $L_{em} > L_g$ if $j \sim .2$. We have already seen from (62,65), once a loop is initially threaded by an external electromagnetic field which is then removed, a current is induced in the loop, trapping a fraction of the flux of order one. Conservation of N ensures that flux is conserved. Now as the loop oscillates, it shrinks; as the radius R_l of the loop decreases (62) tells us J will scale as $J \propto R_l^{-1}$. Thus in theory even if the initial current $J_i \ll J_{max}$, L_{em} eventually dominates L_g. The very low frequency waves emitted by the loop can not propagate through the plasma, the ambient plasma frequency is 13 or so orders of magnitude greater than that of the emitted waves. Thus each loop blows a bubble around it to a final radius $R_s \propto j_i^{\frac{1}{5}}\mu^{\frac{3}{10}}(1 + z_d)^{\frac{-3}{10}}Mpc$[5]. The loops heat up their surroundings, generating large dense shells of gas, the galaxies form on these gravitationally unstable shells with their present distribution representing the distribution of bubbles when fragmentation occurred. The technical details of the calculation can be found in [5]. They find that for

$$\frac{L_{em}}{L_g}\mid_{initially} > 10^{-3},$$

$R_s \sim 10 - 20h^{-1}Mpc$, in agreement with [8]. Dark matter remains inside the voids, if it is cold then it can not catch up the expanding shells of gas, indicating that local dynamical measurements of Ω in the vicinity of galaxies and clusters will be a significant underestimate of the true value. As the loop continues to shrink, $J \rightarrow J_{max}$. The string saturates, producing high energy cosmic rays, charged particles, photons, neutrinos and possibly as yet unknown physics. Bose carriers lead to a first order phase transition and a non superconducting state, at $J_{max} \sim 10^{20}$ amperes for GUT models. In a typical case the energy released electromagnetic radiation is 10^{49} $ergs^{-1}$ or 10^{66} erg in total. When the current saturates, a loop will emit vast amounts of particles. The loop may be seen as a high energy x-ray source at $z \sim 10 - 50$. Such loops will also contribute to the hard x-ray and γ-ray backgrounds, aswel as the 10^{20}ev cosmic rays. One possible problem this scenario faces, is how to obtain the primordial magnetic fields. With $G\mu \sim 10^{-6}$, they require the primordial energy density in magnetic fields $\sim 3.10^{-9}$ of the radiation energy density to induce large enough currents. No natural mechanism exists to generate such large magnetic fields. Random fluctuations in the χ field on the string, will induce a very small initial current as we shall see.

Vilenkin and Field[46] proposed that loops of superconducting cosmic strings which possess cusps emit short bursts of highly directed electromagnetic radiation from these regions, possibly accounting for the jets observed in quasars, formed from accelerated particles. The loops of cosmic strings provide the central engines of the quasars. The calculation is based on the assumption that the mean power radiated from superconducting cosmic strings at current j is, $P \propto j^{\frac{4}{3}}$, and is dominated by short cusp bursts[48,49]. This relationship and the effect of current on cusps will be discussed later. The total power emitted is estimated at $\sim 10^{62}$ erg.

Returning to the action for the superconducting cosmic strings, (68), we will now look at the equation of motion and obtain solutions[37]. Then we will look at the radiation from these solutions and compare the answers with previous estimates[48,49,50]. The variation with respect to θ and x^μ yield

$$\partial_a(\sqrt{-\gamma}\gamma^{ab}j_b) = 0 \qquad\qquad \partial_a(\sqrt{-\gamma}(\gamma^{ab} + \Theta^{ab})\partial_b x^\mu) = 0 \qquad (69)$$

where the world sheet energy momentum tensor is defined by

$$\Theta^{ab} = \frac{-2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{ab}} = 2j^a j^b - \gamma^{ab}(j^2) \tag{70}$$

Choosing the orthonormal gauge, $\gamma_{\sigma\tau} = 0, \gamma_{\sigma\sigma} + \gamma_{\tau\tau} = 0$ then using (65) in (69) we obtain

$$(\partial_\tau^2 - \partial_\sigma^2)\theta = 0 \qquad or \qquad \theta = f(\sigma + \tau) + g(\sigma - \tau) \tag{71}$$

with f and g arbitrary functions. Now the orthonormal gauge is invariant under the coordinate transformation,

$$\sigma + \tau \to \overline{\sigma} + \overline{\tau} = \frac{2}{A} f(\sigma + \tau)$$

$$\sigma - \tau \to \overline{\sigma} - \overline{\tau} = \frac{2}{A} g(\sigma - \tau)$$

so in those coordinates, $\theta = A\overline{\sigma}$ only, provided the current is non-zero everywhere. We have from (65)

$$j_0 = 0, \quad j_1 = -A\sqrt{\frac{K_{eff}}{\mu}} = j \tag{72}$$

and (69) becomes

$$d_\tau[(1 - \gamma^{11}j^2)\partial_\tau x^\mu] = \partial_\sigma[(1 + \gamma^{11}j^2)\partial_\sigma x^\mu] \tag{73}$$

$$\gamma^{11} = [(\partial_\sigma x^\mu)^2]^{-1} \tag{74}$$

Now if $(\partial_\sigma x)^2 + j^2 = 0$, then we have the solution $x^0 = j\tau$, $\underline{x} = \underline{x}(\sigma)$, an arbitrary function subject to the above condition. These arbitrary static curves are called 'springs'[37]. What is happening is that the positive pressure contributed by the current cancels the string tension. This may be seen from the stress tensor

$$\begin{aligned} T^{\mu\nu} &= -2\frac{\delta S}{\delta \eta_{\mu\nu}} \\ &= \mu \int d^2\sigma \sqrt{-\gamma}(\gamma^{ab} + \Theta^{ab})\partial_a x^\mu \partial_b x^\nu \delta^4(x^\mu - x^\mu(\sigma)) \end{aligned} \tag{75}$$

In our gauge, for a loop of large radius of curvature, we approximate it as a straight line in the z- direction, i.e $x^\mu = (\tau, 0, 0, \sigma)$. We then have

$$T^{\mu\nu} = \mu \delta^2(x) diag(1 + j^2, 0, 0, 1 - j^2) \tag{76}$$

This result differs from[55], as mentioned earlier. Really there will be surface terms contributing to T_{rr}, $T_{\theta\theta}$ in the case of the infinite string. For $j \equiv \frac{J}{J_{mas}} = 1$, the negative pressure or string tension term is cancelled, leaving effectively a line of pressureless dust. This is the significance of J_S. It appears that as the current approaches the critical value, springs will form, they behave as matter and will not decay for they have stopped oscillating. This is potentially disasterous and in fact was first mentioned by OTW[5,37]. If the current density is non zero, we can not have 'cusps' in superconducting cosmic strings, for as $\acute{x} \to 0, \gamma^{11} \to \infty$, yet $\gamma^{11}j^2 = 1$ is a gauge condition.

Another class of solutions are oscillating loops which are also solutions to (73)[50]. For $\acute{x}^2 = const$, $x_0 = \tau$, we have,

$$\ddot{\underline{x}} = v^2 \underline{x}''$$

where,

$$v^2 = \frac{1 - \frac{\acute{j}^2}{\acute{x}^2}}{1 + \frac{\acute{j}^2}{\acute{x}^2}}$$

with the general solution

$$\underline{x}(\sigma,\tau) = (\frac{1}{2})[\underline{a}(\sigma + v\tau) + \underline{b}(\sigma - v\tau)] \tag{77}$$

The gauge conditions imply

$$\acute{\underline{a}}^2 = \acute{\underline{b}}^2 = \zeta^2 = \frac{2}{1 + v^2} \tag{78}$$

when we impose the condition $\acute{\underline{a}}.\acute{\underline{b}} = 0$. Thus $\acute{\underline{a}}$ and $\acute{\underline{b}}$ are curves lying on a sphere of radius ζ. They are periodic with zero centre of mass. Solutions of this type have been found for ordinary cosmic strings [49]. They possess kinks, where the $\acute{x}$ derivative is discontinuous. The simplest kinky loop has $\acute{\underline{a}}$ being two points, the north and south pole, $\acute{\underline{b}}$ two antinodal points on the equator. From (77,78) we find $\acute{x}^2 = \frac{1}{1+v^2}$, hence we have

$$v^2 = \frac{\sqrt{(1 + 8j^2)} - (1 + 2j^2)}{2j^2} \tag{79}$$

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The effect of the current has been to reduce the velocity of wave propagation on the string to $v^2 < 1$. As j increases, v decreases until $v = 0$ *when* $j^2 = 1$, the kinks do not propagate around the loop and we are in the realm of springs. The fact the current slows down the kinks is enough to make the power radiated from the oscillating loop finite. Previous estimates at the electromagnetic radiation from loops have assumed they obey a Nambu trajectory. This has led to infinite answers for the total power, hence the introduction of cut offs into the answers[49,48]. Yet we have seen the dramatic effect the current has on the motion of the loop. Using the solution (77,78) we were able to evaluate exactly the power radiated from a kinky loop, and obtained

$$\frac{dP}{d\Omega} \propto j^2(1 - j^2)^2 In(\frac{1}{j})$$

which is finite for both small and large j. In fact the total power radiated is also finite, tending to zero as $j \to 0$, *and as* $j \to 1$. The explanation is simple. It comes from the fact that kinks slow down producing springs as $j \to 1$. The overall result turns out to be far less power than previous estimates gave, making the quasar jet scenario difficult to reconcile with these string solutions.

We have seen a simple argument that radiation from loops and their consequent shrinkage leads to a build up in current. Basically because the total winding number of the θ field is fixed, (barring tunneling events), $j \propto \nabla\theta$. As the loop loses energy it shrinks, therefore $\nabla\theta$ rises. This result is important in the OTW scenario, where whatever the primordial magnetic field ($B_{primordial}$) is, eventually a loop will radiate away lots of electromagnetic radiation. This power, $P_{em} \sim \Gamma_{em}\mu j^2$ dominates the gravitational radiation, $P_g \sim \Gamma_g G\mu^2$ for $j > \sqrt{G\mu} \sim 10^{-3}$. Thus $j \geq 10^{-3}$ for any interesting effects. However when j=1, springs form, so we require $10^{-3} \leq j_c < 1$. We know that as $j \to 1$ the Nambu action acquires corrections, what about the region close to the lower limit? Recall $J_a = 2eK_{eff}\partial_a\theta$. If any region has $\nabla\theta > \nabla\theta_c$, it will turn critical and lose its' current. This translates into if $| \acute{x} |< \frac{|j|}{|j_c|}$ in that region, current will be lost. Now for small j_c the motion of the loop should be nearly Nambu, allowing cusp like regions where $| \acute{x} |\sim 0$. These regions will lose current. Meanwhile the loop shrinks an amount $\Delta L \sim \Gamma_g G\mu L$ every period, L is the length of the loop. Thus there is a current gain due to radiation from the loop and a competition

$$\frac{dj}{dt} |_{loss} = -\frac{j^2}{j_c L}$$

$$\frac{dj}{dt}\big|_{gain} = \Gamma G\mu \frac{j}{L} + \Gamma_{em}\frac{j^3}{L}$$

There exists a stable fixed point at $j \sim 10^{-4}j_c$ *when* $\frac{dj}{dt} = 0$[50]. This value is too small to cause the explosions required, hence would imply large initial currents or primordial magnetic fields, as loop shrinkage appears to lead to current loss, not buildup.

Many of the possible problems with superconducting cosmic strings lie in the observation that they like to form springs. This of course is model dependent, so given the region of parameter space in which superconducting cosmic strings form, what fraction of these solutions will be springs? Many authors have tackled this issue[52,53,51,54], and come up with a wide cross section of conclusions. A common result is that springs can form, but do not have to. However, it appears that if we demand that superconducting cosmic strings remain around long enough to be cosmologically interesting ($K \geq 20$), this constrains the parameters in the Higgs potential, the result being that such loops with long lived currents are generically springs (i.e exist for a wide range of $\frac{\lambda_1}{\lambda_2}$)[51].

Recently solutions have been discussed in which the string possesses charge j_0 aswel as current j_1[57]. These objects are a kind of cross breed between the usual spring solutions and the non-topological soliton solutions, in that the presence of a conserved charge enables the strings to be stabilised for a much lower current than is required to form springs (for couplings $\lambda_3 \gg 1$). These 'vorton' solutions if stable against decay, would behave as matter, and very quickly come to dominate the energy density of the universe, a big problem for the explosion scenario, and for superconducting cosmic strings in general. Again by carefully choosing the coupling constants, like springs, vortons could be avoided. However cries of 'fine tuning' may well then be heard.

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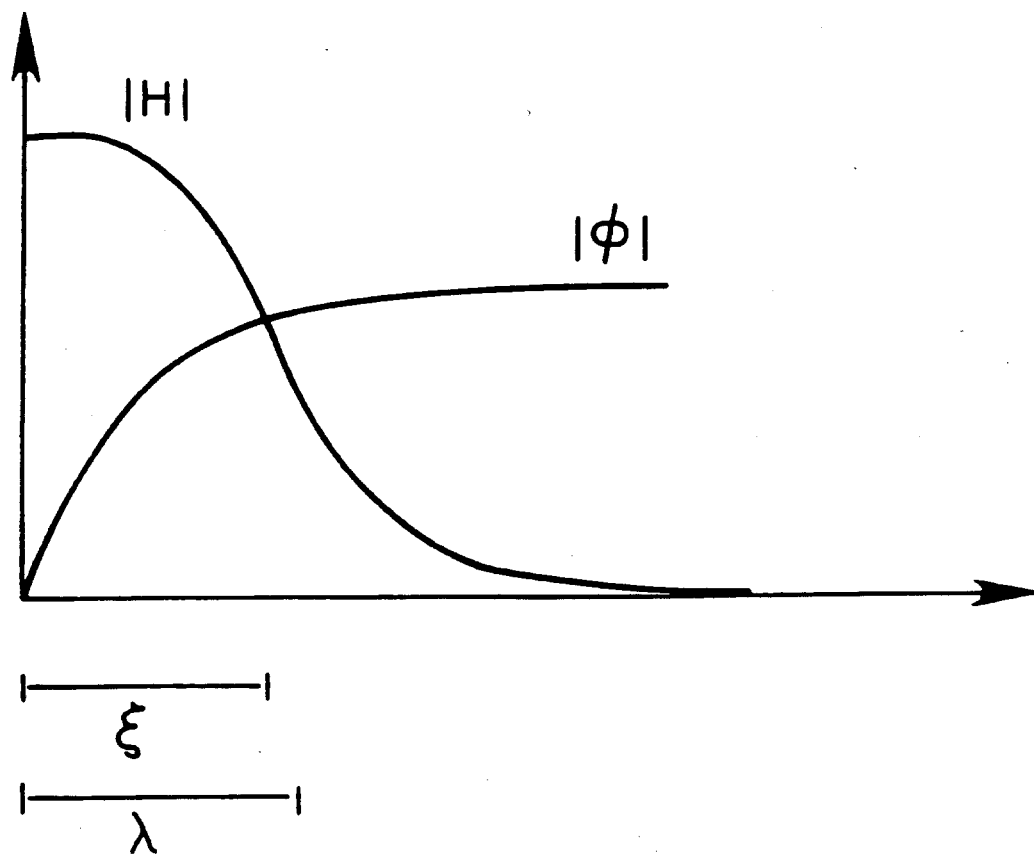[57] R. L. Davis and E. P. S. Shellard, *The physics of vortex superconductivity II* MIT preprint (1988)

Figure-1.
Field configuration for a Nielsen-Olesen vortex solution.